

An Analysis of Adequate Inventory Levels

Abstract: An analytical procedure for determining adequate stock levels for an inventory system with random demand and replenishment functions is presented.

Introduction

This analysis was motivated by the necessity of determining adequate levels of parts inventory for an operating plant. Both the demand for parts and the arrival of new stock for inventory replenishment had a definitely stochastic, or random, character. The problem was to provide a reasonably accurate procedure for determining adequate stock levels.

An introduction to the problem of inventory analysis can be found in the literature.* It should be noted that in most such analyses, inventory levels are "optimized" relative to a cost criterion and on the assumption that demand and/or replenishment are predictable and nonstochastic. This paper treats the situation in which both usage and replenishment of inventory have a random character.

More or less typical analysis of inventory levels proceeds from a consideration of a function of the following type:

$$\text{Total Cost} = T. C. = AI + BI^{-1} + K p(I),$$

where I is the initial level of inventory at the start of an order cycle, $p(I)$ is the expected cost of running out of stock and A and B are appropriate cost factors. In many cases, the factor K is found to be indeterminate or very inaccurate and $K p(I)$ is not included in the analysis. As a substitute, the concept of a protective stock level is introduced. However, in either case an analysis of the probability of running out of stock is an underlying requirement.

The basic problem is the determination of the probability of running out of stock, for any given stock item, as a function of initial inventory level at the start of a given time interval and as determined by appropriate random variables for the demand on and the replenishment of that stock. More specifically, the demand for a given stock item is taken to be a composite of a fluctuating daily demand and another demand which is a random function of time and quantity. The arrival of a new supply or a replenishment of stock is taken to be a random function of time.

Demand analysis

Typical inventory data indicate that the daily demand d for a single day may be represented by a probability density function of the form

$$p_D(d) = 0 \quad d < d_0$$

$$p_D(d) = e^{-\bar{\lambda}} \frac{(\bar{\lambda})^{d-d_0}}{(d-d_0)!} \quad d \geq d_0$$

for appropriate d_0 and $\bar{\lambda}$. The average or expected daily demand $\bar{\lambda}$ is given by $\bar{\lambda} = d_0 + \lambda$. This function is illustrated by Fig. 1.

The demand for t days then has the corresponding function:

$$p_D(d) = 0 \quad d < d_0 t$$

$$p_D(d) = e^{-\bar{\lambda}t} \frac{(\bar{\lambda}t)^{d-d_0t}}{(d-d_0t)!} \quad d \geq d_0 t.$$

Throughout this paper the time $t \geq 0$.

The other demand, called the "extraneous demand," may be represented by a probability density function of the form $\pi_E(k, t) p_E(k, j)$ where $\pi_E(k, t)$ is the probability that exactly k "extraneous" requisitions will be made in t days and $p_E(k, j)$ is the conditional probability that exactly j items will be demanded if k requisitions are made.

If during a replenishment cycle (i.e., the time between two successive arrivals of new stock) the probability that no requisitions will be made by time t is given by e^{-at} , then

$$\pi_E(k, t) = e^{-at} \frac{(at)^k}{k!}.$$

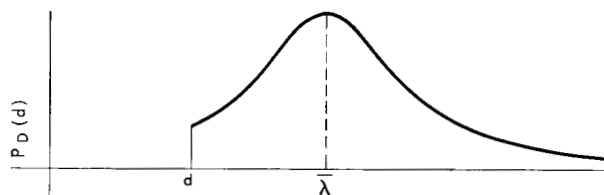


Figure 1

From typical inventory requirements data,

$$p_E(k, j) = 0 \quad \text{for } j < k\mu_0$$

$$p_E(k, j) = e^{-k\tilde{\mu}} \frac{(k\tilde{\mu})^{j-k\mu_0}}{(j-k\mu_0)!} \quad \text{for } j \geq k\mu_0,$$

where the expected value for j is $k(\mu_0 + \tilde{\mu})$. Here μ_0 may be regarded as the minimum extraneous demand.

Now if $p_E^*(j)$ denotes the probability that the extraneous demand in t days will be exactly j items one may write:

$$p_E^*(j) = \sum_{k=0}^{[j/\mu_0]} e^{-at} \frac{(at)^k}{k!} e^{-k\tilde{\mu}} \frac{(k\tilde{\mu})^{j-k\mu_0}}{(j-k\mu_0)!},$$

where $[j/\mu_0]$ denotes the maximum integer $< j/\mu_0$.

In lieu of a neat analytic simplification for $p_E^*(j)$ we use an approximation of the form

$$p_E^*(j) = e^{-\tilde{k}(t)\tilde{\mu}} \frac{(\tilde{k}(t)\tilde{\mu})^{j-\tilde{k}(t)\mu_0}}{(j-\tilde{k}(t)\mu_0)!}.$$

This expresses the probability of exactly j items being demanded in t days in terms of the expected number of demands $\tilde{k}(t)$. From $\pi_E(k, t)$, the expected value of k is $\tilde{k}(t) = at$.

Using the density functions $p_D(d)$, $p_E^*(j)$ and the well-known convolution formula for Poisson-type functions of this kind, one finds that $P_D(d_t \geq D)$, the probability of the total demand in t days being at least D items, can be written as

$$P_D(d_t \geq D) = \sum_{n=[D-At]}^{\infty} e^{-L_t} \frac{(L_t)^n}{n!},$$

where $L = \tilde{\lambda} + a\tilde{\mu}$ $A = d_0 + a\mu_0$

and $[D-At]$ denotes the least integer $> D-At$. In the following development the bracket will be omitted, but implied.

Supply or replenishment analysis

For the replenishment of stock through the arrival of shipments,

$$P_S(t; s \leq \bar{s}) = P_A(t)P_{s,A}(s \leq \bar{s}) + [1 - P_A(t)]P'_{s,A}(s \leq \bar{s})$$

where $P_S(t; s \leq \bar{s})$ is the probability that in t days the number of items arriving is at most $\bar{s}$, and $P_A(t)$ is the probability that at least one shipment arrives in t days. $P_{s,A}(s \leq \bar{s})$ is the conditional probability that the shipment quantity is at most $\bar{s}$ if a shipment arrives. $P'_{s,A}(s \leq \bar{s}) = 1$ is the conditional probability that the shipment quantity is at most $\bar{s}$ if no shipment arrives.

$$\text{We take } P_A(t) = 0 \quad \text{for } t < t_0$$

$$= 1 - e^{-\sigma(t-t_0)} \quad \text{for } t \geq t_0 \geq 0,$$

with appropriate t_0 and

$$P_{s,A}(s \leq \bar{s}) = 0 \quad (\bar{s} < \hat{s} = \text{amount ordered})$$

$$= 1 \quad (\bar{s} \geq \hat{s}).$$

Probability of stock out

If m_t represents the net supply of stock on hand at time t days, then $m_t = I + \bar{s}_t - d_t$ where I is the initial stock supply at $t=0$, $\bar{s}_t$ is the increase in supply through replenishment up to t days and d_t is the demand up to t days. Let $P_N(m_t \leq M)$ be the probability that the net supply at t is at most M units; then

$$P_N(m_t \leq M) = \int_{\bar{s}_t} P_D(d_t \geq I - M + \bar{s}_t) dP_S(t; s \leq \bar{s}_t).$$

If one supposes the replenishment quantity $\bar{s}_t$ is so large that

$$P_D(d_t \geq I - M + \bar{s}) = 0$$

then

$$P_N(m_t \leq M) = P_D(d_t \geq I - M) \quad \text{for } t < t_0$$

$$= e^{-\sigma(t-t_0)} P_D(d_t \geq I - M) \quad \text{for } t \geq t_0.$$

Setting $M=0$ gives $P_N(m_t \leq 0)$, the probability of running out of stock within t days.

Substituting the expression developed for $P_D(d_t \geq I - M)$ in the Demand Analysis section, one obtains

$$P_N(m_t \leq M) = \sum_{n=I-M-At}^{\infty} e^{-L_t} \frac{(L_t)^n}{n!} \quad \text{for } t < t_0$$

$$= e^{-\sigma(t-t_0)} \sum_{n=I-M-At}^{\infty} e^{-L_t} \frac{(L_t)^n}{n!} \quad \text{for } t \geq t_0.$$

In order to determine an adequate level of inventory I , the procedure is to obtain the function

$$\bar{P}_N(I) = \text{Max}_I P_N(m_t \leq 0),$$

from which the value of I required for a certain prescribed value for $\bar{P}_N(I)$ may be obtained. This value of $I = I_0$ is then the minimum level of inventory needed to insure that the probability of a stock out is not more than $\bar{P}_N(I_0)$.

The curve for $\bar{P}_N(I)$ is obtained by calculating for each given value of I the value of $t = \hat{t}$ for which $P_N(m_t \leq 0)$ is a maximum and then by calculating $\bar{P}_N(I) = P_N(m_{\hat{t}} \leq 0)$. The value $\hat{t}$ is determined by using a differentiable approximation to $P_N(m_t \leq 0)$ whose maximum may then be obtained by elementary calculus.

An approximation to $P_D(d_t \geq D)$

In order to evaluate $P_N(m_t \leq 0)$ at $\hat{t}$ it is proposed to replace $P_N(m_t \leq M)$, through approximation, by a more manageable differentiable function. It is sufficient, therefore, to find a differentiable approximation $\psi(t, D)$ for $P_D(d_t \geq D)$. Consideration of the function

$$P_D(d_t \geq D) = \sum_{n=D-At}^{\infty} e^{-L_t} \frac{(L_t)^n}{n!}$$

indicates the following characteristics:

$$\text{For } t=0 \quad P_D(d_0 \geq D) = 0 \quad \text{when } D \geq 1 \quad (1)$$

$$\text{For } t \geq \frac{D}{A} \quad P_D(d_t \geq D) = 1 \quad (2)$$

$$\text{For } t=0 \quad P_D(d_{1/A} \geq D) - P_D(d_0 \geq D) = 0 \quad (3)$$

$$\text{For } t = \frac{D-1}{A} \quad P_D(d_{D/A} \geq D) - P_D(d_{D-1/A} \geq D) = e^{-(L \frac{D-1}{A})} \quad (4)$$

and this $\rightarrow 0$ for practical values of the parameters D, A, L .

$$\text{For } t = t_i = \frac{D}{A+L} \quad \text{i.e. } D - At_i = Lt_i \quad (5)$$

$$P_D(d_i \geq D) = \frac{1}{2}.$$

$$\text{For } t = t_i \quad P_D(d_{i+1/A} \geq D) - P_D(d_i \geq D) = S, \quad (6)$$

where from the Appendix:

$$S = A e^{-L t_i} \frac{(L t_i)^{D - A t_i - 1}}{(D - A t_i - 1)!} \left\{ 1 + \frac{L}{A} \frac{D - A t_i - 1}{L t_i} \right\}$$

$$\text{For } D=0 \quad P_D(d_i \geq 0) = 1 \quad \text{for all } t. \quad (7)$$

$$\text{For } D \rightarrow \infty \quad P_D(d_i \geq D) = 0 \quad \text{for all finite } t. \quad (8)$$

$$\text{For } D = (A+L)t \quad P_D(d_i \geq D) = \frac{1}{2} \quad \text{for any given } t. \quad (9)$$

Taking now a function of the form

$$\psi(t, D) = K(Lt)^m e^{-\gamma(Lt)^r},$$

one can observe that the required conditions (1) and (3) at $t=0$ will be satisfied if m and r are both >0 or <0 . To meet conditions (2) and (4), the zero of

$$\frac{d\psi}{dt} = \psi(t, D) \frac{m - \gamma r (Lt)^{r-1}}{t}$$

is taken at the point $t = \frac{D}{A} = T_D$; that is,

$$\frac{m}{\gamma r} = (LT_D)^r.$$

Thus at T_D (2) is satisfied and condition (4) gives the equation

$$K \left((LT_D)^r e^{-1} \right)^m = 1.$$

From these relationships one obtains

$$\psi(t, D) = \left[\left(\frac{t}{T_D} \right)^r e^{1 - \left(\frac{t}{T_D} \right)^r} \right]^{\frac{m}{r}}.$$

Condition (5) yields

$$\left[\tau^r e^{1 - \tau^r} \right]^{\frac{m}{r}} = \frac{1}{2}, \quad (5')$$

$$\text{where } \tau = \frac{A}{A+L}.$$

Using S as defined in (6) (cf. Appendix) one obtains

$$\frac{m(1 - \tau^r)}{\tau} = 2ST_D. \quad (6')$$

The first of these (5') yields

$$m = \frac{\log \frac{1}{2}}{\log \tau + \frac{1}{r}(1 - \tau^r)},$$

where $\log$ is to base e , here and throughout the following.

Combining this with the (6') one gets

$$\frac{1}{r} + \frac{\log \tau}{1 - \tau^r} = -G \quad (7')$$

$$\text{where } G = -\frac{\log \frac{1}{2}}{2ST_D} = \frac{0.3466(A+L)}{SD}$$

$$= 0.8688 \sqrt{\frac{L}{A+L}} \frac{\sqrt{I}}{I-1} \quad (\text{cf. Appendix}).$$

One can show that a solution for $r > 0$ exists if the parameters I, A, L satisfy

$$0.758 < \frac{IL}{A+L} < 3.032,$$

and a solution for $r < 0$ exists if

$$3.032 < \frac{IL}{A+L}.$$

In practical situations the latter condition prevails. A suitably accurate approximation may be used in expression (7') to calculate r . It is clear from the above that all other required parameters in $\psi(t, D)$ may be obtained from r .

One may also verify (7), (8), (9). This then determines the approximation $\psi(t, D)$ for $P_D(d_i \geq D)$ in the interval $0 \leq t \leq T_D$. For $t > T_D$ one defines the approximating function to $\equiv 1$.

Application of these formulae

Using the approximation $\psi(t, D)$ one considers

$$P_N(m_i \leq M) = \psi(t, I-M) \quad t < t_0 \\ = e^{-\sigma(t-t_0)} \psi(t, I-M) \quad t \geq t_0.$$

The problem then for given I and M is to find the best $t = \hat{t}$ for which $P_N(m_i \leq M)$ is maximal. Evaluation of $P_N(m_i \leq M) = \bar{P}_N$ then yields the probability that the net supply of stock on hand will fall below M units at the worst point of time $\hat{t}$. In order to determine $\hat{t}$ one must consider the critical points $t = t_0$ and $t = T_D$ where the functions are redefined.

If $T_D \leq t_0$ then $\hat{t} = t_0$ and $\bar{P}_N = 1$.

If $T_D > t_0$ one determines the value of $t = t_{\max}$, at which $e^{-\sigma(t-t_0)} \psi(t, I-M)$ has its maximum. If $t_{\max} \leq t_0$ then $\hat{t} = t_0$ and $\bar{P}_N = \psi(t_0, I-M)$; if $t_{\max} > t_0$ then $\hat{t} = t_{\max}$ and

$$\bar{P}_N = e^{-\sigma(t_{\max}-t_0)} \psi(t_{\max}, I-M).$$

In this manner one can compute the desired probability $\bar{P}_N$ for each $I-M$.

Figure 2 presents a curve of $\bar{P}_N$ vs I for a stock item with demand and supply parameter values as follows:

Mean daily demand $\bar{\lambda} = 25$

Minimum daily demand $d_0 = 20$

Mean extra demand $\bar{\mu} = 21$

Minimum extra demand $\mu_0 = 13$

Average number of days between extra demands = 30

Average number of days in replenishment interval = 30

Maximum number of days in replenishment interval = 32.

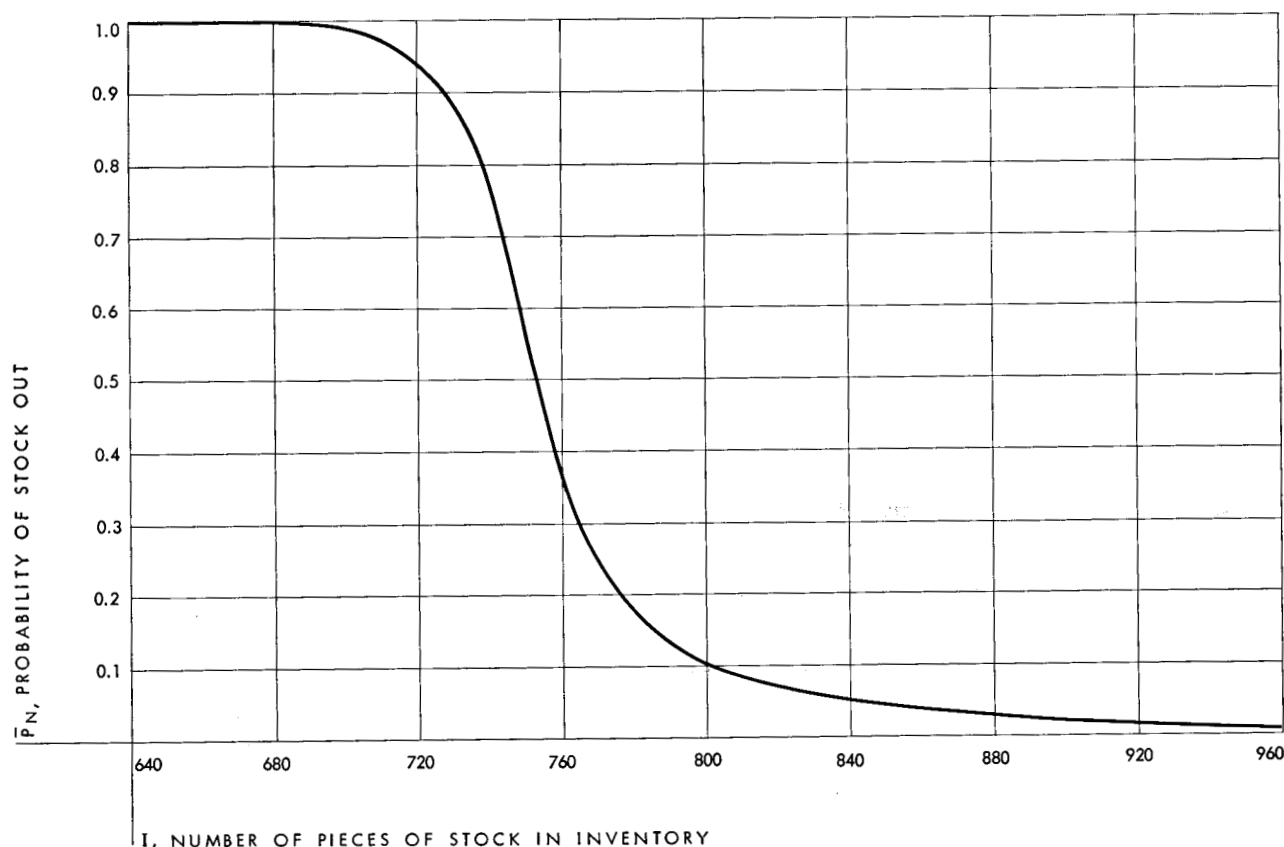


Figure 2 Probability of stock out versus number of pieces of stock in inventory.

Appendix

In order to determine a value for the slope S of $P_D(d_t \geq D)$ consider the difference

$$\Delta P_D = P_D(d_{t+\Delta t} \geq D) - P_D(d_t \geq D)$$

$$= \sum_{D-A_{t-1}} e^{-\left(Lt + \frac{1}{A}\right)} \frac{\left(Lt + \frac{1}{A}\right)^n}{n!} - \sum_{D-A_t} e^{-Lt} \frac{(Lt)^n}{n!}$$

where $\Delta t = \frac{1}{A}$. Writing $g_n(t) = e^{-Lt} \frac{(Lt)^n}{n!}$

$$\begin{aligned} \Delta P_D &= \left[\sum_{D-A_{t-1}} g_n\left(t + \frac{1}{A}\right) - \sum_{D-A_{t-1}} g_n(t) \right] + \left[\sum_{D-A_{t-1}} g_n(t) - \sum_{D-A_t} g_n(t) \right] \\ &= e^{-Lt} \frac{(Lt)^{D-A_{t-1}}}{(D-A_{t-1})!} \left\{ 1 + \frac{L}{A} \frac{D-A_{t-1}}{Lt} \right\}, \end{aligned}$$

and using Stirling's formula for large D one gets when

$$t = t_i = \frac{D}{A+L}$$

$$\Delta P_D = A \frac{1}{\sqrt{2\pi D^*}} \left\{ 1 + \frac{L}{A} \left(1 - \frac{1}{D^*} \right) \right\} \quad \text{where } D^* = \frac{D}{1 + \frac{A}{L}}.$$

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